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# Courant Institute of Mathematical Sciences

MAGNETO-FLUID DYNAMICS DIVISION

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# Asymptotic Equivalence of the Navier-Stokes and Nonlinear Boltzmann Equations

HAROLD GRAD

September 30, 1964

AEC Research and Development Report

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# Abstract

Previous theory of the linear Boltzmann equation is extended to somewhat nonlinear problems. For sufficiently small initial data, solutions are shown to exist for macroscopically long times. If the mean free path is small the nonlinear solution is particularly well approximated by the linear solution, and it is also approximated by the macroscopic Navier-Stokes solution of an equivalent problem.

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## 1. Introduction

One of the most interesting features of the use of Boltzmann's equation to describe the behavior of a gas is the possibility of a radically different macroscopic description of the gas via the Navier-Stokes or inviscid Euler Equations. The fact that there is an overlapping range where both theories are believed to be valid on physical grounds suggests the strictly mathematical question of a connection between the theories of the Boltzmann and Navier-Stokes equations. The relationship is extremely singular, first because the variables used to describe the state of the gas are so dissimilar, and second because the time scales on which the gas evolves are apparently unrelated.

To exhibit this connection turns out to require much more accurate estimates of solutions than those given by any naive proofs of existence. There are strong nonlinear existence theorems in the large for the spatially homogeneous problem. There is one by Carleman [1] for hard spheres, one by Wild [2] modified by Morgenstern [3] for a cutoff Maxwellian potential; and there is an existence theorem for an artificial equation which reduces to a legitimate Boltzmann equation (with a general cutoff potential) in the space-independent case by Povzner [4]. But this space-independent problem, although mathematically substantial, is physically trivial since there is no change at all in any of the macroscopic variables and there is, of course, no fluid dynamics without spatial variation. Previous existence theorems in problems with space dependence are relatively poor.[1] Existence is shown for a time comparable to a collision time; this is quite irrelevant on any macroscopic time scale in which fluid behavior is to be observed. These limitations in the nonlinear spatially-dependent theory are not surprising when one notes that this theory should be expected to be at least as difficult as the corresponding nonlinear theory of the Navier-Stokes equations (which should arise as a specal limiting case).

There are statements in the literature that one should not even expect solutions in the large for the nonlinear Boltzmann equation, and more tractable abstract replacements have therefore been suggested for the Boltzmann equation [7],[4]. Of course, there is no *a priori* reason to expect the existence of solutions in the large for *any* nonlinear equation. This doubt holds just as well (in the absense of any information) for the equations of fluid dynamics. But it is exactly the connection between the Boltzmann equation and fluid dynamics, especially the Navier-Stokes equations, that leads one to expect the liklihood of existence of nonlinear solutions of the Boltzmann equation in the large (at least for sufficiently small mean free path). As a matter of fact, the presence of strong results in the spatially homogeneous case where fluid dynamics is absent suggests that nonlinear estimates similar to those in fluid dynamics are exactly what is missing from the Boltzmann theory.

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[1] See Ref. [5], Chap. III. The posthumous proof of Carleman (Ref. [6], pp. 87-92) is vitally incomplete.

According to some recent usage, the term "Boltzmann equation" is taken to refer to any equation which involves a distribution function in velocity as well as physical space. We shall be more restrictive and consider the original equation of Boltzmann and Maxwell in its application to a classical gas. Even with this restriction, we deal with a general type rather than with a specific equation. A common property of all these equations is that of conservation of mass, momentum, and energy. This is violated, for example, by equations governing a neutron gas. The type of result which is usually adequate in the latter case, viz., exponential decay or exponential growth, requires relatively trivial estimates. In the linear version, the dominant feature is whether the collision operator is positive or not. For a classical gas the operator is non-negative; but the origin belongs to the spectrum as a consequence of conservation. In the spatially homogeneous case the zero eigenvalues can be discarded and we do obtain the uniform exponential decay of all solutions. In the general case we find (but this is a much more subtle result) an approximately exponential decay to a situation in which fluid behavior takes over. The state of the gas then evolves at a much slower rate and ultimately does decay to equilibrium despite the zero eigenvalues.

The variety of true Boltzmann equations is associated with the choice of intermolecular force law. There is at the present time no existence theory whatsoever for any equation with a potential that extends to infinity (infinite total cross-section). This is even true for a Maxwellian potential for which the collision operator is known to have a complete pure point spectrum with explicit polynomial eigenfunctions [8]. The one exception is the spatially homogeneous problem in which the solution is explicit as a sum of exponentially decaying eigenfunctions. Nothing is known about the spectrum for any other non-cutoff potential, although the spectrum is known for a large class of cutoff potentials (and is, in general, not discrete)[9]. Even with cutoff potentials, the mathematical tractability seems to be extremely sensitive to the type of cutoff.[2] But there does exist a simple way of truncating the collision cross-section which has recently been found to be amenable to theory [9]. The question whether the Boltzmann equation for molecules with infinite total cross-section is mathematically sound (even in the linear case) is completely open.

Discouragement with the Boltzmann equation has led to the study of various types of modified equations. One type of modification, guided by mathematical considerations, replaces the collision operator, which is classically an operator in velocity space alone, by an operator which is nonlocal in physical space. This allows proof of nonlinear existence theorems in the large [4],[7]. But these models seem to make any property other than the bare fact of existence even more elusive than originally. For example, the identification with physical parameters is hidden; the collision operator would require re-definition near a boundary; the relation to fluid dynamics is not simple and is, in any case, not that of an ideal gas. And, of course, the basic arbitrariness in the kernel describing the space-averaging is awkward. If the solutions could be shown to be insensitive to the precise degree of averaging in physical space, then a theory of the actual Boltzmann equation would probably ensue. This type of uniform estimate would seem to be comparable in difficulty to the uniform estimates which are required to establish the singular connection with fluid dynamics. At present there are no such uniform estimates, and the future status of these models is uncertain.

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[2] Personal communication from L. Finkelstein.

An entirely different type of modification of the Boltzmann equation, referred to as a relaxation model, is guided by physical principles and an entirely different type of mathematical expediency (e.g. see [10]). These equations exhibit no greater simplicity with regard to general existence theory, but do offer practical advantages of explicit solutions in simple problems and are more attractive for numerical work.

Existence of solutions to the linear Boltzmann equation for hard spheres was shown by Carleman (Ref. [6], Part 2, Chap. I). Although existence was proved in the large, the growth estimate $e^{t/\tau}$ ($\tau$ is the mean collision time) is catastrophically poor on a macroscopic time scale (typically $\tau \sim 10^{-10}$sec.). This linear existence theory has been generalized recently to include a wide class of intermolecular potentials, and it has been improved to show boundedness for all time [9] and even an ultimate decay to equilibrium [11]. Even more important, the bounds have been shown to be independent of the mean free path (which enters as a singular parameter) and therefore carry over to the macroscopic regime. These strong estimates have been used to show the relation between Boltzmann solutions and fluid solutions, making rigorous the Hilbert theory [12]; and in this paper we use the strong linear estimates to derive certain nonlinear existence results and to evaluate the legitimacy of the linear theory.

The connection between the Boltzmann equation and fluid dynamics was shown formally by Hilbert [13].[3] Introducing a singular parameter $\varepsilon$ into the Boltzmann equation ($\varepsilon$ is the mean free path or mean collision time, cf. Sect. 2), Hilbert showed that an expansion of the solution in powers of $\varepsilon$ has the formal property that each term in the expansion is determined by an algorithm involving the fluid state alone. A plausible argument can be given to show that Hilbert's expansion is asymptotic in $\varepsilon$ except for a short initial layer, a narrow boundary layer, and any internal shock layers (Ref. [5], ¶ 26). The linear estimates mentioned above (uniform in $\varepsilon$) provide a proof that the Hilbert expansion, suitably modified, is asymptotic to arbitrary solutions of the linear Boltzmann equation with smooth initial data [12]. Specifically, after an initial transient of duration $O(\varepsilon)$, the distribution function becomes approximately locally Maxwellian. The fluid state is governed by the inviscid Euler equations of fluid dynamics for a time $O(1)$ and by the Navier-Stokes equations for a time $O(1/\varepsilon)$ (ths viscosity coefficient is $O(\varepsilon)$).

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[3] For a critical account of the Hilbert and Chapman-Enskog theories see Ref.[5], Chap.IV; for recent developments see Ref.[12].

In this paper we use the linear estimates to prove existence of solutions to the nonlinear Boltzmann equation for a length of time which depends on the amplitude of the initial data. For sufficiently small initial amplitude, the solution exists for interesting macroscopic times, but it is in general approximated by the linear equation for only a short (microscopic) part of this time. If the mean free path is small in addition, then the linear approximation is valid for much longer macroscopic times. Furthermore, both solutions are approximated by fluid dynamics (inviscid Euler or Navier-Stokes equations depending on the circumstances). The common fluid behavior for small $\varepsilon$ is the reason for the increased range of validity of the linear approximation. The natural time of evolution of fluid properties is macroscopic and bears no relation to the mean collision time, which is the basic time scale for Boltzmann solutions.

The principal mathematical difficulty in the linear theory arises from the fact that the linear collision operator, although non-negative, is not positive definite. But it is exactly this property which provides the fluid behavior in the limit of small $\varepsilon$. There is a rapid approach in a time of order $\varepsilon$ toward a state which is essentially macroscopic. This is followed by fluid behavior, largely inviscid on a finite time scale, but viscously decaying on a long time scale of order $1/\varepsilon$. Without the zero eigenvalues, all solutions would decay to zero in a time comparable to $\varepsilon$ and both the theory and the physical behavior would be relatively trivial.

The principal difficulty in the nonlinear problem is that *a priori* estimates in the linear problem are natural in a norm which is inappropriate for estimation of the nonlinear terms. To find estimates in compatible norms requires the use of explicit smoothing properties of a part of the linear collision operator (after removing a singular and unbounded part) and depends heavily on the assumed rectangular domain and simple boundary conditions. The rectangular domain with perfectly reflecting walls has the special property of eliminating the boundary layer which was mentioned earlier and which requires special estimates. The boundary layer is of course absent in an unbounded domain. But this case requires additional estimates of the decay at infinity in physical space; this problem is treated elsewhere.[4]

## 2. Properties of the Boltzmann Equation[5]

We consider a dimensionless distribution function $F(\xi,x,t)$ of the dimensionless arguments velocity ($\xi$), space($x$), and time ($t$). The variables have been made dimensionless with respect to the parameters of a reference state which is specified by a mass density $\rho_o$ and mean thermal speed $C_o = (RT_o)^{\frac{1}{2}}$. The distribution function has been made dimensionless by a factor $\rho_o/C_o^3$, the arguments $\xi$, $x$, $t$ by respective factors $C_o$, a mean free path $\lambda_o$ (related to $\rho_o$, $C_o$ and molecular constants), and a mean free time $\lambda_o/C_o$. A skeleton form of the Boltzmann equation is

$$DF \equiv \frac{\partial F}{\partial t} + \xi_i \frac{\partial F}{\partial x_i} = Q(F,F) \quad . \tag{2.1}$$

The collision term $Q$ is a quadratic operator on $F$ as a function of $\xi$ alone; the streaming term $DF$ operates on $F$ as a function of $(x,t)$ alone with $\xi$ as a parameter. Initial and boundary values can be expected to be the same as for the differential operator alone.

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[4] To appear in Communications on Pure and Applied Mathematics.

[5] For a more detailed description and proofs of the properties quoted in this section see Ref. [9].

The dimensionless equilibrium solution (Maxwellian) which corresponds to the reference state $(\rho_o, C_o)$ is $F = \omega(\xi)$ where

$$\omega = \frac{1}{(2\pi)^{3/2}} \exp(-\tfrac{1}{2}\xi^2) \quad . \tag{2.2}$$

We have

$$Q(\omega,\omega) = 0 \quad . \tag{2.3}$$

We are interested in solutions of (2.1) which are not greatly perturbed from $F = \omega$; to this end we introduce

$$F = \omega + \omega^{\frac{1}{2}} f \tag{2.4}$$

(the factor $\omega^{\frac{1}{2}}$ is convenient for normalization of f). The equation staisfied by f is

$$DF + LF = \nu\Gamma(f,f) \tag{2.5}$$

where

$$LF = -2\omega^{-\frac{1}{2}} Q(\omega,\omega^{\frac{1}{2}} f) \tag{2.6}$$

$$\nu\Gamma(f,f) = \omega^{-\frac{1}{2}} Q(\omega^{\frac{1}{2}} f,\ \omega^{\frac{1}{2}} f) \tag{2.7}$$

and the function $\nu(\xi)$ will be defined in a moment. The linear operator L is singular and unbounded. However, for a large class of appropriately restricted intermolecular potentials, it can be shown to split into two terms,

$$Lf = \nu f - Kf \quad . \tag{2.8}$$

The singular and unbounded part of L is given by the first term; K is a bounded (even compact) integral operator. The factor $\nu(\xi)$, the collision frequency of a molecule of speed $\xi$, is a certain unbounded positive function of $\xi$. The factor $\nu$ in (2.7) was removed because the remaining factor $\Gamma$ will be found to be bounded in an appropriate sense.

We now quote a number of properties of the operator L which are proved elsewhere [9]; the required properties of $\Gamma$ are derived here in the Appendix. The derivation of these properties is dependent on specific assumptions with regard to the intermolecular force law; mainly that it be repulsive, of finite total cross-section (suitably defined), and sufficiently "hard" (i.e. the force is sufficiently strong for small distances of separation). Classical hard spheres and suitably cut off power law forces with exponent $\geq 5$ are included.

The operators L and K are symmetric and L is non-negative; these statements are made relative to the $L_2$ norm in the infinite three-dimensional space $\xi$,

$$\begin{aligned} ||f||^2 &= \int f^2 d\xi \\ (f,g) &= \int f\, g\, d\xi \quad . \end{aligned} \tag{2.9}$$

A critical point in all the analysis is that L is not positive. The origin is a multiple eigenvalue

$$L\psi_\alpha = 0 \qquad (\alpha = 0,1,2,3,4) \tag{2.11}$$

with the explicit normalized eigenfunctions

$$\begin{cases} \psi_0 = \omega^{\frac{1}{2}} \\ \psi_i = \xi_i \omega^{\frac{1}{2}} \quad (i = 1,2,3) \\ \psi_4 = (\xi^2 - 3)(\omega/6)^{\frac{1}{2}} \quad . \end{cases} \tag{2.11}$$

The existence of these eigenfunctions is closely related to the fact that mass, momentum, and energy are conserved; and it is because L is not positive that solutions of the Boltzmann equation do not decay simply to zero but exhibit fluidlike behavior under appropriate circumstances.

The function $\nu(\xi)$ is usually monotone, it is bounded away from zero (for the class of hard potentials under consideration), and it is usually unbounded for large $\xi$ as a power $\xi^{\gamma}$, $0 \leq \gamma \leq 1$. We shall require only the bounds

$$1 \leq \nu(\xi) < \nu_1(1+\xi^2)^{\frac{1}{2}} \tag{2.12}$$

where $\nu_1$ is a constant. The lower bound has been taken to be 1 by a suitable choice of the normalizing dimensional mean collision time.

Since K is compact, the essential spectrum of L consists of exactly the values taken by $\nu(\xi)$. Therefore the origin is an isolated point of the spectrum, and we have

$$\begin{aligned} &(f,\ Lf) > \mu(f,f) \\ &(f,\ \psi_\alpha) = 0 \end{aligned} \tag{2.13}$$

for some positive constant $\mu$.

We denote by $\kappa_o$ the bound of K,

$$\|Kf\| \leq \kappa_o \|f\| \quad . \tag{2.14}$$

Other more refined properties of K are

$$\max |Kf| < \kappa_1 \|f\| \tag{2.15}$$

and

$$\max(1+\xi^2)^{(r+1)/2}|Kf| < \kappa_{r+2}\max(1+\xi^2)^{r/2}|f| \tag{2.16}$$

$$(r = 0, 1, \ldots).$$

In other words, Kf is bounded if f is square integrable, and Kf decays faster than f by a factor $\xi$ for large $\xi$.

There are several alternate forms of the Boltzmann equation that we shall use. If, instead of (2.4), we introduce

$$F = \omega + \varepsilon\omega^{\frac{1}{2}}f \tag{2.17}$$

into the Boltzmann equation (2.1), we obtain

$$Df + Lf = \varepsilon\nu\Gamma(f,f) \quad . \tag{2.18}$$

If $\varepsilon$ is small, we are led to consider the linear equation

$$Df + Lf = 0 \tag{2.19}$$

with the expectation that solutions of this equation will approximate those of (2.18) for small $\varepsilon$.

A more interesting change of scale is

$$F = \omega/\varepsilon + \omega^{\frac{1}{2}}f \tag{2.20}$$

which leads to the equation

$$Df + \frac{1}{\varepsilon} Lf = \nu \Gamma(f,f) \ . \tag{2.21}$$

Implicit in (2.20) is the supposition that the perturbation is not only small relative to the reference state $\omega/\varepsilon$, but that the perturbation has a basic length scale which is long compared to the mean free path of the reference state [which is $O(\varepsilon)$ since the density is $O(1/\varepsilon)$]. It is apparent that $\varepsilon = 0$ is a singularity. Moreover, it is not immediately evident that solutions of (2.21) will be approximated by those of the associated linear equation

$$Df + \frac{1}{\varepsilon} Lf = 0 \ . \tag{2.22}$$

Despite appearances, it will turn out that solutions of the linear equation (2.22) do approximate those of the nonlinear equation (2.21). The reason is their common fluidlike behavior in the singular neighborhood of small $\varepsilon$.

A final change of scale which we shall wish to consider is

$$F = \omega/\varepsilon + \varepsilon\omega^{\frac{1}{2}} f \tag{2.23}$$

which leads to the equation

$$Df + \frac{1}{\varepsilon} Lf = \varepsilon\nu \Gamma(f,f) \ . \tag{2.24}$$

This equation is "linearized" to a higher order ($\varepsilon^2$) than the mean free path is shortened ($\varepsilon$). It is associated with the same linear equation as (2.21), viz. (2.22), but the relation is, in this case, more evident.

Implicit in the various scalings in $\varepsilon$ is the assumption that the initial data are independent of $\varepsilon$.

## 3. Boundary Conditions

We consider a rectangular domain and two types of boundary conditions, periodicity and specular reflection. The simpler case is periodicity. We take a fundamental domain with sides $(a_1, a_2, a_3)$ and assume that the initial values are periodic in $x_i$ with periods $a_i$. It is easily verified that this periodicity is retained by all functions that arise in the analysis, in particular by the solution. The periodicity property is used essentially at only one point in the analysis, to show that a boundary integral vanishes in making an *a priori* estimate.

The specular reflection boundary condition, also applied in a rectangular domain, states that f is an even function of the normal component of velocity at a wall

$$f(\xi) = f(\xi - 2n(\xi \cdot n)) . \tag{3.1}$$

By reflection of the fundamental domain D with respect to each of three coordinate planes we obtain a domain D* consisting of eight replicas of D. In D* the function f satisfies a periodicity boundary condition. It is easy to verify that the symmetry of the initial function f with respect to the three planes which bisect D* is maintained by all functions which arise in the analysis. Thus solution of the Boltzmann equation with a periodic boundary condition in D* yields the desired solution with specular boundary condition in the subdomain D.

There are a few precautions that should be observed in passing between D and D*. If the initial function f is required to be continuous in D*, this implies that the boundary condition (3.1) must be satisfied by the initial function in D. A continuous initial derivative $\partial f/\partial x_1$ in D* implies the condition

$$\frac{\partial}{\partial x_1} f(\xi_1) = - \frac{\partial}{\partial x_1} f(-\xi_1) \tag{3.2}$$

at the wall perpendicular to $x_1$, etc. In other words, smoothness in D involves satisfaction of certain relations beyond conventional smoothness at the boundary. These relations can be interpreted as smoothness conditions as seen by a given particle which is specularly reflected (and changes its velocity discontinuously while doing so).

The problems of initial data which do not satisfy (3.1) or (3.2), of more general discontinuous initial data, and of general nonrectangular domains are closely related but will not be considered here.

## 4. Norms

In order to properly combine estimates for the linear and nonlinear equations it is necessary to introcude an assortment of norms. We denote by N the $L_2$ norm in $(\xi, x)$ (as distinguished from $||\ldots||$, previously used for the norm in $\xi$)

$$N\,[f] = \left\{ \int f^2 d\xi \; dx \right\}^{\frac{1}{2}} \quad . \tag{4.1}$$

The simple maximum norm we denote by

$$N_o[f] = \max_{\xi, x} |f| \tag{4.2}$$

and more generally

$$N_r[f] = \max_{\xi, x} (1+\xi^2)^{r/2} |f| \quad . \tag{4.3}$$

We shall also use the combined norms

$$\not{N}_r[f] = \max_{\xi} (1+\xi^2)^{r/2} \left\{ \int f^2 dx \right\}^{\frac{1}{2}} \tag{4.4}$$

and the norms of x-derivatives

$$N^s[f] = \max_{n=o}^{s} N\,[\nabla^n f] \tag{4.5}$$

$$N_r^s[f] = \max_{n=o}^{s} N_r[\nabla^n f] \tag{4.6}$$

$$\not{N}_r^s[f] = \max_{n=o}^{s} \not{N}_r[\nabla^n f] \tag{4.7}$$

where

$$|\nabla^n f|^2 = \sum_{i_1 \ldots i_n} \left| \frac{\partial^n f}{\partial x_{i_1} \ldots \partial x_{i_n}} \right|^2 \tag{4.8}$$

We shall use the symbol $\eta$ to represent any one of the norms $N$, $N_r^s$, $\mathbb{N}_r^s$. Each of the norms is evaluated at a given time, t. For the maximum with respect to t we use

$$\eta = \max_t \eta(t) \tag{4.9}$$

where max is understood to be taken for $0 < t < t_o$, $t_o$ finite or infinite depending on the context.

Some elementary estimates are

$$N_r^s \leq N_{r+1}^s, \qquad N_r^s \leq N_r^{s+1} \tag{4.10}$$

$$\mathbb{N}_r^s \leq \mathbb{N}_{r+1}^s, \qquad \mathbb{N}_r^s \leq \mathbb{N}_r^{s+1} \tag{4.11}$$

$$N \leq \pi \mathbb{N}_2 \tag{4.12}$$

the last being a consequence of the fact that $(1+\xi^2)^{-1}$ is square integrable $[\int (1+\xi^2)^{-2} d\xi = \pi^2]$. There exists a constant $s_o$ depending on the domain such that (Sobolev)

$$N_r^s < s_o \, \mathbb{N}_r^{s+2}. \tag{4.13}$$

From Sect. 2 we transfer the $L_2$ inequalities (2.13) and (2.14) unaltered to the space $(\xi, x)$ and generalize (2.15) and (2.16) to read

$$\mathbb{N}_o^s[Kf] < \kappa_1 N^s[f] \tag{4.14}$$

$$\mathbb{N}_{r+1}^s[Kf] < \kappa_{r+2} \mathbb{N}_r^s[f] \; . \tag{4.15}$$

Similar estimates for the quadratic collision term $\Gamma$, derived in the Appendix, are

$$N_r[\Gamma(f,g)] \leq \gamma_o N_r[f] N_r[g] \tag{4.16}$$

and

$$\bar{N}_r[\Gamma(f,g)] \leq \gamma_o \bar{N}_r[f] N_r[g] \ . \tag{4.17}$$

There is no estimate of $\Gamma$ in terms of only $L_2$ estimates of f and g.

The above estimates of K and $\Gamma$ form the basis for the entire theory. We continue with some consequences of these estimates.

From $|\xi \cdot \nabla f| < (1+\xi^2)^{\frac{1}{2}} |\nabla f|$ we obtain

$$\bar{N}_r^s[\xi \cdot \nabla f] < N_{r+1}^{s+1}[f] \tag{4.18}$$

and from (2.12) we have

$$\bar{N}_r^s[\nu f] < \nu_1 \bar{N}_{r+1}^s[f] \tag{4.19}$$

from which follows

$$\bar{N}_r^s[Lf] < (\nu_1 + \kappa_{r+1}) \bar{N}_{r+1}^s[f] \tag{4.20}$$

using (2.8), (4.15), and (4.11). If f is a sufficiently regular solution of the linear Boltzmann equation (2.19), $\partial f/\partial t$ can be estimated from (4.18) and (4.20) as

$$\bar{N}_r^s[\partial f/\partial t] < (1 + \nu_1 + \kappa_{r+1}) \bar{N}_{r+1}^{s+1}[f] \ . \tag{4.21}$$

A vital property of all norms is

$$\mathcal{N}[\tilde{f}] \leq \mathcal{N}[f] \tag{4.22}$$

where

$$\tilde{f}(\xi,x,t) = f(\xi,x-s\xi,t) \tag{4.23}$$

and s is a constant (note that this would not be true if $\mathcal{N}$ were replaced by the stronger norm with $\max_\xi$ preceding $\int \ldots dx$).

The basic formula to which all convergence estimates will be referred is the

Lemma: If

$$f(\xi,x,t) = \frac{\nu}{\varepsilon}\int_0^t e^{-\nu(t-s)/\varepsilon}\, g(\xi,x-\xi(t-s),s)ds \tag{4.24}$$

then

$$\mathcal{N}_r[f] \leq \overline{\mathcal{N}}_r[g] \ . \tag{4.25}$$

If $\nu$ is a constant, this estimate for any norm is classical since f is a mean value with parameter s and total weight less than one of the functions g,

$$(\nu/\varepsilon)\int_0^t e^{-\nu(t-s)/\varepsilon} = 1 - e^{-\nu t/\varepsilon} < 1 \ .$$

With $\nu$ a function of $\xi$, the estimate (4.25) is not true for the $L_2$ norm of N and a maximum norm with respect to $\xi$ is needed. For the proof, we use Schwartz's inequality to obtain

$$\int g(\xi,x-\xi(t-s_1),s_1)g(\xi,x-\xi(t-s_2),s_2)dx$$

$$\leq (1+\xi^2)^{-r}(\overline{\mathcal{N}}_r[g])^2$$

which implies

$$\int f^2dx \leq (1+\xi^2)^{-r}(\overline{\mathcal{N}}_r[g])^2(1-e^{-\nu t/\varepsilon})^2 \ ,$$

and thus (4.25).

## 5. Outline of the Proof

The linear equation (2.22) can be written

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla f + \frac{1}{\varepsilon} \nu f = \frac{1}{\varepsilon} Kf \tag{5.1}$$

and transformed into the integral equation

$$f(\xi,x,t) = f_o(\xi,x-\xi t)\exp(-\nu t/\varepsilon) + \frac{1}{\varepsilon}\int_0^t \exp[-\nu(t-s)/\varepsilon]k(\xi,x-\xi(t-s),s)ds \tag{5.2}$$

where

$$k = Kf \tag{5.3}$$

and

$$f(\xi,x,0) = f_o(\xi,x) \tag{5.4}$$

is a given initial function. We solve (5.1) by iteration on the right side, or the equivalent, iterate (5.2) and (5.3) successively. The crude estimate obtained by setting the exponentials in (5.2) equal to one yields convergent iterations with a growth $\exp(\kappa t/\varepsilon)$ in any norm $\mathcal{N}$ where $\kappa$ is the bound of K in the particular norm $\mathcal{N}$. No great improvement in this growth estimate can be made by examining the iterates more carefully (unless $\nu$ is a constant, see below) because of the nontrivial nullspace of the operator L; Kf is "just as large" as $\nu f$.

But for any smooth solution f, we can obtain the a priori estimate[6]

$$N[f] \leq N[f_o] \ . \tag{5.5}$$

Multiplying through the equation (5.1) by f and integrating with respect to x and $\xi$, we can drop the streaming term $\nabla \cdot (\frac{1}{2} \xi f^2)$ using periodicity in x and estimate the collisional term by $(f, Lf) \geq 0$, to obtain (5.5). A simple argument shows that (5.5) obtains without unnecessary smoothness. In only the case $\nu$ = constant (cutoff Maxwellian molecules) can the uniform bound on N[f] be established directly for each iteration (the $L_2$ norm of K is just $\nu$). If $\nu$ is a function of $\xi$, we are blocked by the inapplicability of the Lemma, (4.25), in an $L_2$ norm.

The mollifying properties of K, (2.15) and (2.16), can now be used to show boundedness of the solution as a function of $\xi$, and a Sobolev lemma to show boundedness as a function of x for sufficiently smooth data (all x-derivatives of f satisfy the same equation and boundary conditions and therefore the same a priori estimate). Note that all these uniform-in-time estimates are also independent of the value of $\varepsilon$.

Similar estimates, also independent of $\varepsilon$, can be found for the inhomogeneous linear equation. Using them, we iterate on the right side of the nonlinear equation (2.21). The unbounded factor $\nu$ on the right is taken care of by the strong inhomogeneous linear estimate which gives a solution decaying faster in $\xi$ than the inhomogeneous term. A more serious problem is to obtain maximum estimates which, for the linear equation, are obtained from the a priori $L_2$ estimates only after losing derivatives. This is taken care of by using a norm $\not{N}^3_f$. Writing $f_i = \partial f/\partial x_i$, $f_{ij} = \partial^2 f/\partial x_i \partial x_j$, etc., we have

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[6] This is essentially the linearized version of Boltzmann's H-theorem.

$$\left\{\begin{aligned} &\frac{\partial}{\partial x_i} \Gamma(f,f) = 2\Gamma(f,f_i) \\ &\frac{\partial^2}{\partial x_i \partial x_j} \Gamma(f,f) = 2\Gamma(f,f_{ij}) + 2\Gamma(f_i,f_j) \\ &\frac{\partial^3}{\partial x_i \partial x_j \partial x_k} \Gamma(f,f) = 2\Gamma(f,f_{ijk}) + 2\Gamma(f_i,f_{jk}) \\ &\qquad\qquad + 2\Gamma(f_j,f_{ki}) + 2\Gamma(f_k,f_{ij}) \,. \end{aligned}\right. \tag{5.6}$$

An estimate for $\not{N}^3_r[f]$, using Sobolev, implies a maximum estimate $N^1_r[f]$. Each term on the right side in (5.6) has at least one argument of $\Gamma$ which is f itself or a first derivative; i.e. at least one argument is bounded and the other one is $L_2$. Thus $\not{N}^3_r[\Gamma(f,f)]$ can be estimated in terms of $\not{N}^3_r[f]$. In this way existence is proved for (2.18)

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla f + Lf = \varepsilon\nu\Gamma(f,f) \tag{5.7}$$

for a time of order $1/\varepsilon$ and for (2.21)

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla f + \frac{1}{\varepsilon} Lf = \nu\Gamma(f,f) \tag{5.8}$$

for a time of order 1. The two results are equivalent in that each describes existence for a long time of order $1/\varepsilon$ times the mean collision time. But the class of initial data in (5.8) is more special in that the $f_o$ is not only small but it is also slowly varying in space compared to the mean free path. We may obtain (5.8) directly from (5.7) by a change of scale, viz., by inserting $f(\varepsilon x)$ and changing the time scale to $\varepsilon t$. In return for the greater specialization of initial values in (5.8), we find that the solution is more accurately linear. In (5.7) the solution is approximated by the corresponding linear solution (2.19) to within an error $O(\varepsilon)$ microscopic time $O(1)$; this is evident by inspection of the nonlinear equation. But a more powerful estimate of the inhomogeneous linear equation can be made in (5.8) and yields the unexpected result that the solution of (5.8) is approximated by the corresponding linear solution (2.22) for the entire (macroscopic) time of existence. The proof of this result is closely related to the proof that the solution of (5.8) is approximated by fluid dynamics. Since the parameter $\varepsilon$ does not appear in the inviscid fluid equations, it is not surprising that a small fluid perturbation of relative size $\varepsilon$ will remain accurately linear for the macroscopic time $O(1)$. For the "more linear" equation (2.24), existence and approximate linearity as well as fluid behavior are proved for a time $O(1/\varepsilon)$. The comparisons with fluid dynamics are contained in an earlier paper [12], but we shall sketch the proofs here for completeness.

More efficient use of the same estimates that show that the solutions of (5.8) are approximately linear would give an improved existence proof for (5.8) for the longer time $O(\log 1/\varepsilon)$, but we shall not bother with this.

## 6. The Linear Equation[7]

Consider first the integral formulation (5.2), (5.3) with an initial function $f_o(\xi,x)$ which is periodic and bounded with respect to one of the norms $\mathcal{N}$. Iterating and estimating $|f|$ by setting $\nu = 0$,

$$\begin{aligned} &\mathcal{N}[f^{n+1}] \leq \mathcal{N}[f_o] + \frac{1}{\varepsilon}\int_0^t \mathcal{N}[k^n(s)]ds \\ &[k^n] \leq \kappa\,\mathcal{N}[f^n] \end{aligned} \qquad (6.1)$$

we obtain a contracting sequence of iterates $\exp(-\kappa t/\varepsilon)f^n$ in the norm $\mathcal{N}$ where $\kappa$ is the bound on the operator K ($\kappa = \kappa_o$ for $\mathcal{N} = N^s$, $\kappa = \kappa_{2+r}$ for $\mathcal{N} = N^s_r$ or $\cancel{N}^s_r$). We may choose to look at (5.1) with $\nu = 0$ as a dominating equation.

The function $f(\xi,x,t)$ defined by the contracting sequence is in the space given by the completion of $\mathcal{N}$ and satisfies (5.2), (5.3). This solution is unique in the class of all functions $f(\xi,x,t)$ which have a finite norm $\mathcal{N}$ for some interval $0 \leq t \leq t_o$; and at any finite t, there is continuous dependence of f on $f_o$ in the norm $\mathcal{N}$.

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[7] The theory of this section is taken from Ref.[9]. We repeat the proofs (in a slightly different form) to make the abstract existence theory self contained (i.e. starting from stated properties of the collision operators).

The solution has the same boundedness and differentiability properties inherent in the norm $\mathcal{N}$ as the initial function. Let us first assume that $N[f_o]$ is finite. From (5.2) we deduce the existence of the directional derivative Df, and in this generalized sense f satisfies the original equation (5.1). If $N_1^1[f_o]$ is finite, then $\partial f/\partial t$, $\xi \cdot \nabla f$, and $\nu f$ exist individually in (5.1) (as measurable $L_2$ functions). If the maximum norm $N_r^s[f_o]$ is finite and $f_o$ as well as the indicated derivatives are continuous, then the solution is continuous together with its derivatives by the uniform convergence in the maximum norm.

Now consider a fixed initial function $f_o$ with finite norm N. We can approximate $f_o$ by a sequence of functions $f_o^{(n)}$ with continuous first derivatives and finite (but not uniformly bounded) $N_3^1\,[f_o^{(n)}]$ such that $N[f_o - f_o^{(n)}] \rightarrow 0$. By the continuous dependence on initial data $N[f^{(n)}] \rightarrow N[f]$ at any time t. But each of the smooth solutions $f^{(n)}$ satisfies the a priori estimate $N[f^{(n)}] \leq N[f_o^{(n)}]$. This follows by multiplication through the Boltzmann equation (which is satisfied pointwise) by $f^{(n)}$, noting that $\xi \cdot \nabla (f^{(n)})^2$ is integrable in $\xi$ and continuous. Consequently f itself satisfies the estimate

$$N[f] \leq N[f_o] \ . \tag{6.2}$$

Exactly the same argument holds for any x-derivatives of f which are square integrable,

$$N^s[f] \leq N^s[f_o] \; . \tag{6.3}$$

For the linear theory alone (cf. Ref. [9]), it is not necessary to be so parsimonious of derivatives in deriving the a priori estimate; but it is vital for the nonlinear application.

To extend these uniform $L_2$ estimates to maximum estimates, we turn to the integral equation (5.2). Referring to the lemma of Sect. 4 and recalling that $\nu \geq 1$, from (5.2) we have

$$\text{₦}^s_j\,[f] \leq \text{₦}^s_j\,[f_o] + \bar{\text{₦}}^s_j\,[k] \; . \tag{6.4}$$

Using (4.14), (4.15) in (5.3), we find

$$\begin{aligned} \text{₦}^s_o\,[k] &< \kappa_1 N^s[f] \\ \text{₦}^s_j\,[k] &< \kappa_{j+1} \text{₦}^s_{j-1}[f] \; , \end{aligned} \tag{6.5}$$

and inserting these in (6.4) we have

$$\begin{aligned} \text{₦}^s_o\,[f] &\leq \text{₦}^s_o\,[f_o] + \kappa_1 \bar{N}^s[f] \\ \text{₦}^s_j\,[f] &\leq \text{₦}^s_j\,[f_o] + \kappa_{j+1} \text{₦}^s_{j-1}[f] \; . \end{aligned} \tag{6.6}$$

Solving these recurrence inequalities for successive j and inserting the a priori estimate (6.3),

$$\text{₦}^s_o\,[f] \leq \text{₦}^s_o\,[f_o] + \kappa_1 N^s[f_o]$$

$$\begin{aligned} \text{₦}^s_1\,[f] &\leq \text{₦}^s_1\,[f_o] + \kappa_2 \text{₦}^s_o\,[f_o] + \kappa_2\kappa_1 N^s[f_o] \\ &\leq (1+\kappa_2)\text{₦}^s_1\,[f_o] + \kappa_2\kappa_1 N^s[f_o] \end{aligned}$$

$$\text{₦}_2^S\,[f] \le (1 + \kappa_3 + \kappa_3\kappa_2)\text{₦}_2^S\,[f_o] + \kappa_3\kappa_2\kappa_1 N^S[f_o]$$

$$\le (1 + \kappa_3 + \kappa_3\kappa_2 + \pi\kappa_3\kappa_2\kappa_1)\text{₦}_2^S\,[f_o]$$

and for $r \ge 2$

$$\text{₦}_r^S\,[f] \le a_r \text{₦}_r^S\,[f_o]\ , \qquad r \ge 2 \tag{6.7}$$

where $a_r$ depends on $\kappa_1 \ldots \kappa_{r+1}$ only. These estimates are distinguished from the $L_2$ estimates (6.3) in that the constants $a_r$ are necessarily larger than one.

For the inhomogeneous equation

$$Df + \frac{1}{\varepsilon}\,\nu f = \frac{1}{\varepsilon}\,Kf + \nu g\ , \qquad f_o = 0 \tag{6.8}$$

we proceed similarly. Iterating

$$\begin{aligned} f(\xi,x,t) &= \int_0^t \exp[-\nu(t-s)/\varepsilon]\ \Phi\,(\xi,x-\xi(t-s),s)ds \\ \Phi &= \frac{1}{\varepsilon}\,Kf + \nu g \end{aligned} \tag{6.9}$$

we obtain existence of f over any time interval in which the norm of $\nu g$ is finite. As an a priori estimate, using the fact that L is non-negative, we have

$$N^S[f] \le \int_0^{t_o} N^S[\nu g]dt \le t_o \overline{N}^S[\nu g] \le \pi\nu_1 t_o \overline{\text{₦}}_3^S\,[g]\ . \tag{6.10}$$

We obtain maximum estimates from the integral expression (6.9) by use of the Lemma (4.25)

$$\begin{aligned} \text{₦}_o^S\,[f] &\le \kappa_1 \overline{N}^S[f] + \varepsilon\overline{\text{₦}}_o^S\,[g] \\ \text{₦}_j^S\,[f] &\le \kappa_{j+1}\overline{N}_{j-1}^S[f] + \varepsilon\overline{\text{₦}}_j^S\,[g] \end{aligned} \tag{6.11}$$

Iterating these inequalities and taking $r \geq 3$, we find

$$\mathcal{N}_r^s\,[f] \leq (b_r t_o + \varepsilon c_r)\bar{\mathcal{N}}_r^s\,[g]\ , \qquad r \geq 3\ . \tag{6.12}$$

Summarizing the essential results we have

Theorem 1: There is a unique solution to

$$Df + \frac{1}{\varepsilon}\,Lf = \nu g$$

for $\mathcal{N}_r^s\,[f_o]$ and $\mathcal{N}_r^s\,[g]$ finite and $r \geq 3$ which satisfies

$$\mathcal{N}_r^s\,[f] \leq a_r \mathcal{N}_r^s\,[f_o] + (b_r t_o + \varepsilon c_r)\bar{\mathcal{N}}_r^s\,[g]\ . \tag{6.13}$$

The constants $a_r$, $b_r$, $c_r$ depend only on $\nu_1$, $\kappa_1 \ldots \kappa_{r+1}$, and, in particular, are independent of $\varepsilon$ and $t$.

Better estimates can be made for both the homogeneous and inhomogeneous equations if $f_o$ is space-independent. Existence follows from the general case. We write the solution

$$\frac{\partial f}{\partial t} + \frac{1}{\varepsilon}\,Lf = 0 \tag{6.14}$$

as the sum of its projection into the nullspace of L and the orthogonal complement;

$$\begin{aligned} f &= \hat{f} + \bar{f} \\ \hat{f} &= \sum_\alpha \rho_\alpha \psi_\alpha\ , \qquad \rho_\alpha = (f,\psi_\alpha) \\ (\bar{f},\psi_\alpha) &= 0\ . \end{aligned} \tag{6.15}$$

The two components satisfy

$$\begin{cases} \dfrac{\partial \hat{f}}{\partial t} = 0 \\[2ex] \dfrac{\partial \bar{f}}{\partial t} + \dfrac{1}{\varepsilon} L\bar{f} = 0 \end{cases} \tag{6.16}$$

The fluid part, $\hat{f}$, is constant in time. From (2.13), we derive

$$||\bar{f}|| \leq \exp(-\mu t/\varepsilon)\, ||\bar{f}_o|| \tag{6.17}$$

and conclude that $\bar{f}$ decays exponentially to zero in a time of order $\varepsilon$. Exactly as before, we extend this estimate to

$$N_r^S[\bar{f}] \leq a_r \exp(-\mu t/\varepsilon) N_r^S[\bar{f}_o] \;, \quad r \geq 2 \tag{6.18}$$

with the same constant $a_r$ as previously.

Next we consider an inhomogeneous term which is orthogonal to $\psi_\alpha$

$$\begin{gathered} \frac{\partial f}{\partial t} + \frac{1}{\varepsilon} Lf = \nu g \;, \qquad f_o = 0 \\ (\nu g, \psi_\alpha) = 0 \end{gathered} \tag{6.19}$$

The same decomposition (6.15) yields $\hat{f} = 0$ and the a priori estimate

$$\begin{aligned} N^S[f] &\leq \int_0^t \exp(-\mu(t-s)/\varepsilon) N^S[\nu g(s)]ds \\ &< \frac{\varepsilon}{\mu} \bar{N}^S[\nu g] \;. \end{aligned} \tag{6.20}$$

This is improved into a maximum estimate exactly as in the space-dependent case, but we note that $\varepsilon/\mu$ takes the place of $t_o$ in (6.10) (the time $t_o$ is still implicit in $\bar{N}^S$);

$$N_r^s\,[f] < (b_r/\mu + c_r)\varepsilon\,\bar{N}_r^s\,[g]\ , \qquad r \geq 3 \tag{6.21}$$

We see that the solution is smaller than the inhomogeneous term by a factor $\varepsilon$ in this case

We summarize the principle results in

Theorem 2: The solution to the space-independent equation

$$\begin{cases} \dfrac{\partial f}{\partial t} + \dfrac{1}{\varepsilon}\,Lf = \nu g \\ \\ (\nu g,\ \psi_\alpha) = 0\ , \qquad (f_o,\ \psi_\alpha) = 0 \end{cases} \tag{6.22}$$

for finite $N_r^s\,[g]$ and $N_r^s\,[f_o]$, $r \geq 3$, satisfies

$$\begin{aligned} N_r^s\,[f] \leq a_r \exp(-\mu t/\varepsilon) N_r^s\,[f_o] \\ + (b_r/\mu + c_r)\varepsilon\,\bar{N}_r^s\,[g]\ . \end{aligned} \tag{6.23}$$

If $x$ occurs as a parameter in $f_o$, we merely replace $N_r^s$ by $\bar{N}_r^s$ in this theorem.

Theorem 2 allows us to improve Theorem 1 in the space-dependent case if $\nu g$ and $f_o$ are orthogonal to $\psi_\alpha$. We write

$$f = f^1 + f^2 \tag{6.24}$$

where

$$\frac{\partial f^1}{\partial t} + \frac{1}{\varepsilon}\,Lf^1 = \nu g\ , \qquad f_o^1 = f_o \tag{6.25}$$

$$Df^2 + \frac{1}{\varepsilon}\,Lf^2 = -\,\xi\cdot\nabla f^1\ , \qquad f_o^2 = 0\ . \tag{6.26}$$

Applying Theorem 2 to (6.25), we have

$$\mathcal{N}_r^s\,[f^1] \leq a_r \exp(-\mu t/\varepsilon)\mathcal{N}_r^s\,[f_o] \\ + (b_r/\mu + c_r)\,\varepsilon\,\bar{\mathcal{N}}_r^s\,[g]\ . \tag{6.27}$$

Noting that $\mathcal{N}_{r-1}^{s-1}\,[\xi\cdot\nabla f^1] \leq \mathcal{N}_r^s\,[f^1]$, Theorem 1 applied to (6.26) yields

$$\mathcal{N}_{r-1}^{s-1}\,[f^2] \leq (b_r t_o + \varepsilon c_r)\ \bar{\mathcal{N}}_r^s\,[f^1] \tag{6.28}$$

Combining these estimates, we have

Theorem 3: The solution to

$$Df + \frac{1}{\varepsilon}\,Lf = \nu g \\ (\nu g,\ \psi_\alpha) = 0\ ,\qquad (f_o,\ \psi_\alpha) = 0 \tag{6.29}$$

satisfies the estimate, for $r \geq 3$,

$$\mathcal{N}_{r-1}^{s-1}\,[f] \leq (1 + b_r t_o + \varepsilon c_r)\Big\{a_r \exp(-\mu t/\varepsilon)\ \mathcal{N}_r^s\,[f_o] \\ + (b_r/\mu + c_r)\,\varepsilon\,\bar{\mathcal{N}}_r^s\,[g]\Big\}\ . \tag{6.30}$$

If $\nu g$ and $f_o$ are orthogonal to $\psi_\alpha$, the solution rapidly decays to a magnitude of order $\varepsilon$; but to obtain this strong estimate, we lose a derivative of $g$ and a factor $(1+\xi^2)^{\frac{1}{2}}$.

## 7. The Nonlinear Equation

We consider the domain and boundary conditions as before and initial functions with a finite value of $\not{N}_3^3\,[f_o]$. We look for a solution of (2.21) by iteration on the nonlinear term,

$$Df^n + \frac{1}{\varepsilon}\,Lf^n = \nu g^{n-1} \tag{7.1}$$

$$g^{n-1} = \Gamma(f^{n-1}, f^{n-1}) \ . \tag{7.2}$$

First we estimate $\not{N}_3^3\,[\Gamma]$ from (5.6). For this purpose it is convenient to introduce the auxiliary norms

$$\begin{aligned}
\tilde{N}_r[\nabla f] &= \Big\{\sum_i (N_r[f_i])^2\Big\}^{\frac{1}{2}} \\
\tilde{N}_r[\nabla^2 f] &= \Big\{\sum_{ij} (N_r[f_{ij}])^2\Big\}^{\frac{1}{2}} \\
\tilde{N}_r[\nabla^3 f] &= \Big\{\sum_{ijk} (N_r[f_{ijk}])^2\Big\}^{\frac{1}{2}}
\end{aligned} \tag{7.3}$$

and similarly for $\tilde{\not{N}}_r$. We note the elementary inequalities

$$\begin{aligned}
N_r[\nabla^s f] &\le \tilde{N}_r[\nabla^s f] \le 3^{s/2}\, N_r[\nabla^s f] \\
\not{N}_r[\nabla^s f] &\le \tilde{\not{N}}_r[\nabla^s f] \le 3^{s/2}\, \not{N}_r[\nabla^s f] \ .
\end{aligned} \tag{7.4}$$

Taking norms in (5.6) and recalling (4.16) and (4.17),

$$\begin{aligned}
\not{N}_3[\nabla\Gamma] \le \tilde{\not{N}}_3[\nabla\Gamma] &\le 2\gamma_o N_3[f]\ \tilde{\not{N}}_3[\nabla f] \\
&\le 2\sqrt{3}\ \gamma_o s_o (\not{N}_3^3\,[f])^2
\end{aligned}$$

$$\begin{aligned}
\not{N}_3[\nabla^2\Gamma] &\le 2\gamma_o \Big\{N_3[f]\ \tilde{\not{N}}_3[\nabla^2 f] + \tilde{N}_3[\nabla f]\ \tilde{\not{N}}_3[\nabla f]\Big\} \\
&\le 12\gamma_o s_o (\not{N}_3^3\,[f])^2
\end{aligned}$$

$$\mathbb{N}_3[\nabla^3\Gamma] \le 2\gamma_o \quad N_3[f]\,\tilde{\mathbb{N}}_3[\nabla^3 f] + 3\tilde{N}\,[\nabla f]\,\tilde{\mathbb{N}}_3[\nabla^2 f]$$

$$\le 24\sqrt{3}\,\gamma_o s_o(\mathbb{N}^3_3\,[f])^2 \ ,$$

we obtain

$$\mathbb{N}^3_3\,[\Gamma(f,f)] \le \gamma_1(\mathbb{N}^3_3\,[f])^2$$

$$\gamma_1 = 24\sqrt{3}\,\gamma_o s_o \quad . \tag{7.5}$$

Similarly, we derive

$$\mathbb{N}^3_3\,[\Gamma(f,g)] \le \gamma_1\,\mathbb{N}^3_3\,[f]\,\mathbb{N}^3_3\,[g] \quad . \tag{7.6}$$

From Theorem 1 applied to (7.1) we have

$$\mathbb{N}^3_3\,[f^n] \le a_3\mathbb{N}^3_3\,[f_o] + t_1\bar{\mathbb{N}}^3_3\,[g^{n-1}]$$

$$t_1 = b_3 t_o + \varepsilon c_3 \ . \tag{7.7}$$

The iteration is contracting if we choose $\mathbb{N}^3_3\,[f_o]$ sufficiently small. Specifically, if we take

$$4a_3\gamma_1 t_1\,\mathbb{N}^3_3\,[f_o] < 1 \ , \tag{7.8}$$

then all iterates are bounded

$$\mathbb{N}^3_3\,[f^n] < 2a_3\,\mathbb{N}^3_3\,[f_o] < 1/(2\gamma_1 t_1) \quad . \tag{7.9}$$

Taking differences in (7.1) and (7.2)

$$\mathbb{N}^3_3\,[f^{n+1} - f^n] \le t_1\mathbb{N}^3_3\,[g^n - g^{n-1}]$$

$$\mathbb{N}^3_3\,[g^n - g^{n-1}] \le \gamma_1\mathbb{N}^3_3\,[f^n + f^{n-1}]\,\mathbb{N}^3_3\,[f^n - f^{n-1}]$$

$$< \frac{1}{t_1}\,\mathbb{N}^3_3\,[f^n - f^{n-1}] \ ,$$

and we see that the iterations contract. From the explicit expression (7.7) for $t_1$, we conclude that (7.8) will be satisfied if $t_o$ is sufficiently small, viz.,

$$t_o \leq \frac{\sigma}{\mathcal{N}_3^3\,[f_o]} - \varepsilon\tau \tag{7.10}$$

where $\sigma = (4a_3b_3\gamma_1)^{-1}$ and $\tau = c_3/b_3$ depend only on the collision operators L and $\Gamma$ and the domain (through $s_o$). A bounded solution exists for $0 < t < t_o$ if $t_o$ and $\mathcal{N}_3^3\,[f_o]$ are appropriately related as in (7.10). Uniqueness and continuous dependence on $f_o$ follow in the usual manner. We state

Theorem 4: A unique solution to the Boltzmann equation

$$Df + \frac{1}{\varepsilon}\,Lf = \nu\Gamma(f,f)$$

exists for a time $t_o$ which is related to the initial norm $\mathcal{N}_3^3\,[f_o]$ by (7.10). The solution satisfies the bound (7.9), $\mathcal{N}_3^3\,[f] < 2a_3\mathcal{N}_3^3\,[f_o]$, over this period of time.

Setting $\varepsilon = 1$ in this theorem yields

Corollary 1: A solution to

$$Df + Lf = \nu\Gamma(f,f)$$

satisfying the bound $\mathcal{N}_3^3\,[f] < 2a_3\mathcal{N}_3^3\,[f_o]$ exists for a time $t_o \leq \sigma/\mathcal{N}_3^3\,[f_o] - \tau$.

Similarly, we obtain

Corollary 2: A solution to

$$Df + Lf = \varepsilon\nu\Gamma(f,f)$$

exists for $t < t_o/\varepsilon$ where $t_o$ is exactly as given in (7.10). This result can be obtained directly from the theorem by the substitutions $\varepsilon \longrightarrow 1$, $\gamma_1 \longrightarrow \varepsilon\gamma_1$.

A further corollary we distinguish as

Theorem 5: A unique bounded solution to

$$Df + \frac{1}{\varepsilon} Lf = \varepsilon\nu\Gamma(f,f)$$

exists for $t < t_o = \sigma/\varepsilon\cancel{N}_3^3[f_o] - \varepsilon\tau$ .

This is obtained from Theorem 4 by the substitution $\gamma_1 \longrightarrow \varepsilon\gamma_1$.

All four results can be summarized by the statement that a solution exists for a sufficiently small initial perturbation from equilibrium; and the length of time for which existence is proved, measured in collision times, is approximately inversely proportional to the relative size of the initial perturbation. But Theorems 4 and 5 are distinguished from the corollaries to Theorem 4 by their closer relation to the linear equation. If $f^*$ is the solution of the linear equation

$$Df^* + \frac{1}{\varepsilon} Lf^* = 0 \;, \quad f_o^* = f_o \;, \tag{7.11}$$

the difference

$$\phi = f - f^* \tag{7.12}$$

satisfies

$$D\phi + \frac{1}{\varepsilon} L\phi = \varepsilon^i\nu\Gamma(f,f), \quad \phi_o = 0 \tag{7.13}$$

where $i = 0$ or $1$ under the conditions of Theorems 4 and 5 respectively. From Theorem 3, recalling that $\nu|$ is orthogonal to $\psi_\alpha$, we conclude that

$$\bar{N}_2^2[\phi] < 4\varepsilon^{i+1}(1 + b_3 t_o + \varepsilon c_3)(b_3/\mu + c_3)\gamma_1 a_3^2 (\bar{N}_3^3\,[f_o])^2. \quad (7.14)$$

The right side is of order $\varepsilon$ for the total time of existence referred to in Theorems 4 and 5. In Theorem 4, $i = 0$ and $t_o = O(1)$, while in Theorem 5, $i = 1$ and $t_o = O(1/\varepsilon)$. Thus $f$ differs from $f^*$ by an error of order $\varepsilon$ over the entire time for which the existence of $f$ is known. Under the conditions of Corollaries 1 and 2, the related linear solution can only be shown to approximate the nonlinear solution for a time which is short compared to the proved time of existence; [$o(1)$ in Corollary 1 and $O(1)$ in Corollary 2]. In the one case (Theorems 4 and 5) the linearization is valid for a time long compared to a mean collision time; in the other case (corollaries) only for a time comparable to a collision time.

The precise estimates of Theorem 3 could be used in the nonlinear existence proof to improve Theorems 4 and 5 (but not the corollaries). This improvement is only by a factor $\log 1/\varepsilon$; i.e. existence can be shown for $t_o = O(\log 1/\varepsilon)$ under Theorem 4 and $(1/\varepsilon)\log 1/\varepsilon$ under Theorem 5. The improvement is so slight because the accurate estimate of Theorem 3 is obtained at a cost of reduced differentiability.

In summary, we note that although the convergence proof as given is not sufficiently delicate to distinguish those cases (Theorems 4 and 5) where the mean free path of the unperturbed equilibrium $\omega/\varepsilon$ is small compared to the length scale of variation of the perturbation f, this distinction can be made by use of Theorem 3.

We conclude this section with a brief description of the existence theorems which can be obtained by these methods for the space-independent problem (although this is not our primary purpose). First, since x does not appear, we can replace the constant $\gamma_1$ in (7.5) by $\gamma_o$. Second we may set $\varepsilon = 1$ since this only alters the time scale. We obtain a solution of

$$\frac{\partial f}{\partial t} + Lf = \nu g \qquad g = \Gamma(f,f) \tag{7.15}$$

by iteration using (6.23)

$$N_3[f] \leq (b_3/\mu + c_3)\overline{N}_r[g] \tag{7.16}$$

and (4.16)

$$N_3[\Gamma(f_1,f_2)] \leq \gamma_o N_3[f_1]N_3[f_2] \ . \tag{7.17}$$

We find that a solution exists for all time and staisfies the bound

$$N_3[f] \leq 2a_3 N_3[f_o] \tag{7.18}$$

if $N_3[f_o]$ is taken sufficiently small, specifically

$$2\gamma_o(b_3/\mu + c_3)a_3N_3[f_o] < 1 \ . \tag{7.19}$$

By a slight modification of this analysis we can show that these solutions are not only bounded but decay exponentially to zero. First we assume in the separation $F = \omega + \omega^{\frac{1}{2}} f$ that $\omega$ is chosen with the parameters of the ultimate equilibrium (they are simply determined by the total mass and energy as given by $f_o$). Then $(f,\psi_\alpha) = 0$ and the approach to equilibrium consists in showing $f \longrightarrow 0$. If we assume that $N[e^{\alpha t} vg]$ is bounded uniformly in t, $0 < \alpha < \mu$, then (6.20) can be improved to read (with $\varepsilon = 1$)

$$N[e^{\alpha t} f] < \frac{1}{\mu-\alpha} \bar{N} [e^{\alpha t} vg] \tag{7.20}$$

whence (6.23) becomes

$$\begin{aligned} N_r[e^{\alpha t} f] &< a_3 \exp(-(\mu-\alpha)t) N_3[f_o] \\ &\quad + (b_3/(\mu-\alpha) + c_3)\bar{N}_3[g] \ . \end{aligned} \tag{7.21}$$

Exactly the same procedures as before yield the result that the solution to (7.15) satisfies the bound

$$N_3[f] \leq 2a_3 N_3[f_o] \exp(-\alpha t) \tag{7.22}$$

provided that

$$2\gamma_o\left(\frac{b_3}{\mu-\alpha} + c_3\right) N_3[f_o] < 1 \quad . \tag{7.23}$$

From (7.23) we see that the decay exponent $\alpha$ can be taken arbitrarily close to $\mu$ if the initial value is small. This is not surprising since the equation is, in this case, approximately linear, and $\mu$ is the distance of the spectrum from the origin.

For any finite initial value, the ultimate decay exponent for large time is arbitrarily close to $\mu$ since $N_3[f]$ becomes small ultimately (7.22), and the theorem can then be reapplied with a new choice of $\alpha$ as in (7.23).

These spatially homogeneous results can be compared to others of Carleman [1] (also [6], Part 1), Wild [2] and Morgenstern [3], and Povzner [4]. Carleman's result is for hard spheres and an isotropic as well as homogeneous f; Wild and Morgenstern consider only the cutoff Maxwellian ($\nu$ = constant); Povzner's equation is comparable to ours in generality. Carleman's class of initial functions $f_o$ is somewhat larger; Morgenstern's and Povzner's initial functions are much more general. On the other hand, Povzner and Morgenstern prove nothing about boundedness or the approach to equilibrium, and Carleman proves a relatively weak form of the approach to equilibrium.

## 8. Comparison with Fluid Dynamics

That the solution of the nonlinear Boltzmann equation is approximated by a fluid flow when the mean free path is small is a consequence of known results for the linear equation (Ref. [12]) using the close correspondence between the linear and nonlinear equations that has just been shown. For completeness of this presentation we shall give a brief discussion of this theory.

Under the conditions of Theorem 4 we can show that the Boltzmann solution is approximated within $O(\varepsilon)$ by a solution of the linear inviscid Euler equations. The viscosity coefficient is of order $\varepsilon$; th s viscosity can be ignored to this order for a finite time. Under the conditions of Theorem 5, the Boltzmann solution is approximated within $O(\varepsilon)$ by a solution of the linear Navier-Stokes equation for the whole time for which existence is shown. Viscosity cannot be ignored for such times of order $1/\varepsilon$.

The Hilbert theory, in its linear version, expands the solution of (2.22) as a power series

$$f_H = f^0 + \varepsilon f^1 + \dots , \qquad (8.1)$$

despite the fact that $\varepsilon = 0$ is singular. The successive terms are determined by an algorithm which involves solution of an integral equation for the $\xi$-dependence and of a linear fluid-dynamic Euler equation (for $f^0$) and inhomogeneous Euler equations (later terms) for the $(x,t)$-dependence. The Hilbert series (8.1) is uniquely determined term by term, by the initial fluid state (also assumed to be a formal power series). In other words, to a given fluid state the Hilbert algorithm assigns a unique distribution function $f_H$. The truncated series (8.1) satisfies an inhomogeneous Boltzmann equation with an inhomogeneous term that has been estimated and shown to be $O(\varepsilon^n)$ in an appropriate norm for any finite time $0 < t < t_0$. The series is therefore asymptotic to a true solution of the Boltzmann equation which has the same initial values.

The initial values that can be met by this procedure are very special. To generalize the Hilbert theory a different formal expansion

$$f_{\mu}(\xi,x,\varepsilon t) = f^{o} + \varepsilon f^{1} + \ldots \tag{8.2}$$

can be found which also satisfies the Boltzmann equation term by term. But, whereas $f_H$ is uniquely determined by the fluid state, $f_{\mu}$ is uniquely determined by its orthogonal complement, and each term in the expansion of $f_{\mu}$ decays uniformly as $\exp(-\mu t/\varepsilon)$. Although $f_H$ and $f_{\mu}$ are determined by the projections of $f_o$ into the nullspace of L and its complement, the initial $f_H$ and $f_{\mu}$ are not just the projections of $f_o$. By summing, $f = f_H + f_{\mu}$, essentially arbitrary initial data can be matched, and we can find an appropriate $f_H$ (satisfying a suitably modified initial fluid state) which is asymptotic, after a short transient, to general smooth solutions of the linear Boltzmann equation.

The Chapman-Enskog procedure is an alternative to Hilbert, one which uses the same terms (8.1) as functions of $\xi$ but inserts a modified $(x,t)$ dependence. Instead of Euler and inhomogeneous Euler equations, the fluid parameters in $f_H$ are determined by Euler, Navier-Stokes, and successively highly order differential equations.

This expansion is also asymptotic to a solution of the Boltzmann equation provided that one can suitably estimate the solutions to the fluid equations. The Hilbert theory has the advantage that it can be carried to arbitrarily high order in $\varepsilon$ by solving hyperbolic differential equations which have a known theory. For the Chapman-Enskog procedure there is no theory beyond the second step (Navier-Stokes). But the (linear) Navier-Stokes solution can be bounded uniformly for all time, whereas the equivalent Hilbert solution grows linearly with time. Thus the second Chapman-Enskog term is asymptotic uniformly in time; for our application we need its validity for $0 < t < t_o/\varepsilon$.

We shall sketch the proofs that the first order Hilbert approximation is asymptotic to the linear solution under conditions appropriate to Theorem 4, $t = O(1)$, and the second order Chapman-Enskog solution is asymptotic to the linear solution under conditions appropriate to Theorem 5, $t = O(1/\varepsilon)$; for more complete results see Ref. [12].

We take an initial function $f_o$ of suitable regularity and construct the solution f of the linear equation

$$Df + \frac{1}{\varepsilon} Lf = 0 \quad . \tag{8.3}$$

For the first case, covered by Theorem 4, we construct a locally Maxwellian comparison function

$$f_E = \sum_\alpha \rho_\alpha(x,t)\psi_\alpha(\xi) \quad . \tag{8.4}$$

The $\rho_\alpha$ are related to the (linearized, dimensionless) nominal fluid state by

$$\begin{cases} \rho_0 = \rho \\ \rho_i = u_i \quad (i = 1,2,3) \\ \rho_4 = (3/2)^{\frac{1}{2}} T \quad . \end{cases} \tag{8.5}$$

We specify the $\rho_\alpha$ in $f_E$ as functions of x and t by choosing them to be solutions of the (linearized, dimensionless) Euler equations

$$\begin{cases} \frac{\partial \rho}{\partial t} + \operatorname{div} u = 0 \\ \frac{\partial u_i}{\partial t} + \frac{\partial \rho}{\partial x_i} + \frac{\partial T}{\partial x_i} \\ \frac{3}{2}\frac{\partial T}{\partial t} + \operatorname{div} u = 0 \end{cases} \tag{8.6}$$

with initial data taken from $f_o$

$$\rho_\alpha(x,0) = (\psi_\alpha, f_o) \quad . \tag{8.7}$$

The difference

$$\phi = f - f_E \tag{8.8}$$

satisfies the equation

$$D\phi + \frac{1}{\varepsilon} L\phi = - Df_E \ . \tag{8.9}$$

By the choice of initial data we have

$$(\phi_o, \psi_\alpha) = 0 \tag{8.10}$$

and, from the fact that $\rho_\alpha$ in $f_E$ satisfies the Euler equations, we can easily verify that

$$(Df_E, \psi_\alpha) = 0 \ . \tag{8.11}$$

Inspection of Theorem 3 shows that, after a short exponential transient, $\phi$ becomes small of order $\varepsilon$. We note that it is the same estimate (Theorem 3) which shows that the linear solution approximates the Euler solution $f_E$ and also that the nonlinear solution approximates the linear solution.

For the second case, covered by Theorem 5, we choose a slightly more complicated comparison function

$$f_N = \hat{f}_N + \varepsilon \bar{f}_N \quad . \tag{8.12}$$

As before, $\hat{f}_N$ is locally Maxwellian

$$\hat{f}_N = \sum_\alpha \rho_\alpha \psi_\alpha \tag{8.13}$$

but the $\rho_\alpha$ are now taken to satisfy the linear Navier-Stokes equations

$$\left\{ \begin{array}{l} \frac{\partial \rho}{\partial t} + \operatorname{div} u = 0 \\ \frac{\partial u_i}{\partial t} + \frac{\partial \rho}{\partial x_i} + \frac{\partial T}{\partial x_i} = \varepsilon\eta[\Delta u_i + \frac{1}{3} \frac{\partial}{\partial x_i} (\operatorname{div} u)] \\ \frac{3}{2} \frac{\partial T}{\partial t} + \operatorname{div} u = \varepsilon \lambda \Delta T \end{array} \right. \tag{8.14}$$

($\varepsilon\eta$ and $\varepsilon\lambda$ are the dimensionless viscosity and heat conductivity coefficients). By inspection of (8.12) one should expect $f_N$ to approximate $f$ to within $O(\varepsilon^2)$. This is indeed possible for a time which is $O(1)$. To accomplish this the $\rho_\alpha$ in $f_N$ would have to be assigned appropriate initial values which are obtained from $f_o$ by a rather complex computation (see Ref. [12]). But, since we are interested in times of order $1/\varepsilon$, we shall only be able to show that $f_N$ approximates $f$ to within $O(\varepsilon)$. This, as it turns out, will permit us to use the simpler initial values (8.7). Summarizing, we specify the leading term $\hat{f}_N$ by (8.13) where the $\rho_\alpha$ are solutions of the Navier-Stokes equations with given initial data (8.7).

The second term in the comparison function $f_N$ is obtained by the Chapman-Enskog algorithm. Formal substitution of

$$\phi = f - f_N \tag{8.15}$$

into the Boltzmann equation yields

$$D\phi + \frac{1}{\varepsilon} L\phi = - Df_N - L\bar{f}_N \tag{8.16}$$

We would like to specify $\bar{f}_N$ to satisfy

$$L\bar{f}_N + D\hat{f}_N = 0 \tag{8.17}$$

since this would make the inhomogeneous term in (8.16) of order $\varepsilon$. This is an integral equation for $\bar{f}_N$, with $D\hat{f}_N$ a known inhomogeneous term. But the nontrivial nullspace of $L$ requires a compatibility condition on $D\hat{f}_N$ which is not satisfied. From our earlier result (8.11) we know that compatibility would be satisfied if $\rho_\alpha$ satisfied the Euler equations. Inspecting

$$D\hat{f}_N = \sum_\alpha \psi_\alpha(\partial\rho_\alpha/\partial t + \xi\cdot\nabla\rho_\alpha) \tag{8.18}$$

and replacing $\partial\rho_\alpha/\partial t$ by their Navier-Stokes equivalents from (8.14) we can set

$$D\hat{f}_N = \delta_o + \varepsilon\delta_1 \tag{8.19}$$

where in $\delta_o$ we collect the Euler terms from (8.14) and in $\delta_1$ the viscous and heat conducting terms. It is easily verified exactly as in (8.11), that

$$(\psi_\alpha, \delta_o) = 0 \tag{8.20}$$

We therefore choose $\bar{f}_N$ not as the (nonexistent) solution of (8.17), but as the unique solution of

$$L\bar{f}_N + \delta_o = 0 \tag{8.21}$$

This leaves a formal remainder in (8.16) of order $\varepsilon$,

$$D\phi + \frac{1}{\varepsilon} L\phi = - \varepsilon D\bar{f}_N - \varepsilon\delta_1 \ . \tag{8.22}$$

We note that the separation (8.19) is strictly formal; the $\rho_\alpha$ as solutions of the Navier-Stokes equations are complicated functions of $\varepsilon$ and, in general, are non-expandable. But the remainder in (8.22) can be estimated to be $O(\varepsilon)$, given sufficiently smooth initial data for the Navier-Stokes solution.

The solution of (8.21) for $\bar{f}_N$ is an explicit linear combination of velocity and temperature gradients with coefficients which are functions of $\xi$ obtained by solution of the integral equation (which can be transformed into a Fredholm equation with compact kernel). It is easily verified that

$$(\psi_\alpha, Df_N) = 0 \quad . \tag{8.23}$$

This is essentially the statement that $\rho_\alpha$ satisfy the Navier-Stokes equations [cf. the missing orthogonality condition for (8.17), $(\psi_\alpha, D\hat{f}_N) \neq 0$]; this is, in fact, the kinetic derivation of the Navier-Stokes equations. We are now in a position to refer again to Theorem 3. Since the inhomogeneous term in (8.22) is of order $\varepsilon$ (uniformly for all time by Navier-Stokes estimates) and is orthogonal to $\psi_\alpha$; and since the initial value $\phi_0$ is orthogonal to $\psi_\alpha$, we conclude that $\phi$ is $O(\varepsilon)$ for $t = O(1/\varepsilon)$ [we do not obtain $O(\varepsilon^2)$ because of the factor $t_0 = O(1/\varepsilon)$ in (6.30)].[8]

The solutions of the nonlinear Boltzmann equation obtained in the previous section have been shown to approximate fluid behavior only to the extent that they are also linear. While these fluid comparisons can be improved to some extent, they cannot be greatly extended by our present methods. For example, the existence theory is basically unsuited to describe a finite strength shock wave. A shock wave of finite amplitude would have a thickness $O(\varepsilon)$ and would require handling estimates $N[\nabla f] = O(\varepsilon^{-1/2})$, $N[\nabla^2 f] = O(\varepsilon^{-3/2})$, etc. A Sobolev estimate would not be strong enough to rule out large excursions in f itself.

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[8] The stronger result quoted in Ref.[12](p.171) is in error. Although the linear Navier-Stokes solution in a bounded domain decays exponentially, the decay is slow as $\exp(-\varepsilon t)$. Terms such as $\int_0^\infty f\,dt$ will produce a factor $1/\varepsilon$ just as it arises from $t_0 \sim 1/\varepsilon$.

Under Theorem 4, a fully developed shock wave with strength of order $\varepsilon$ would have a finite thickness, $O(1)$. Uncer Theorem 5, with nonlinearity of order $\varepsilon^2$, the fully-developed thickness would be $O(1/\varepsilon)$. Thus weak shocks, and even much steeper profiles, are contained within the theory. However, the time required to observe the development of a steady limiting shock profile would be $O(1/\varepsilon)$ under Theorem 4 and $O(1/\varepsilon^2)$ under Theorem 5. We could therefore observe only the initial stages of shock formation.

One may hope that stronger nonlinear results for the Boltzmann equation will be obtained by making efficient use of nonlinear Navier-Stokes estimates.

## Appendix: Estimate of the Quadratic Collision Integral[9]

In a more complete notation, the term Q in (2.1) can be written

$$Q(F,F) = \int (F'F_1' - FF_1)d\Omega \qquad \text{(A.1)}$$
$$d\Omega = B(\theta,V)d\theta d\varepsilon d\xi_1 \;.$$

Introducing f as in (2.4), we obtain the bilinear form

$$\nu\Gamma(f,g) = \int \tfrac{1}{2}(f'g_1' + f_1'g_1 - fg_1 - f_1g)\omega_1^{\frac{1}{2}}d\Omega \qquad \text{(A.2)}$$

where

$$\nu = \int \omega_1 d\Omega \;. \qquad \text{(A.3)}$$

To estimate $\Gamma$ we write

$$|\nu\Gamma| \leq \nu\Gamma_1 + \nu\Gamma_2$$
$$\nu\Gamma_1 = \int \tfrac{1}{2}|fg_1 + f_1g|\omega_1^{\frac{1}{2}}d\Omega \qquad \text{(A.4)}$$
$$\nu\Gamma_2 = \int \tfrac{1}{2}|f'g_1' + f_1'g_1|\omega_1^{\frac{1}{2}}d\Omega \;.$$

To estimate $\Gamma_1$ we need the bound

$$\sigma = \int \omega_1^{\frac{1}{2}}d\Omega < \alpha_o\nu \qquad \text{(A.5)}$$

for some constant $\alpha_o$. This is easily obtained from the explicit forms

$$\nu(\xi) = 2\pi \int B_o(|\eta-\xi|)\omega(\eta)d\eta \qquad \text{(A.6)}$$
$$\sigma(\xi) = 2\pi \int B_o(|\eta-\xi|)\omega^{\frac{1}{2}}(\eta)d\eta$$

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[9] For the notation, transformations, and supporting estimates used in this Appendix, we refer to Ref. [9].

where

$$B_o(V) = \int B_o(\theta,V)d\theta \quad . \tag{A.7}$$

The bound (A.5) follows from (A.6) as an elementary exercise if, for example, $B_o(V)$ has a uniformly bounded derivative (this is true for all hard power law potentials).

From

$$|f| < (1+\xi^2)^{-r/2}\ N_r[f] < N_r[f]$$

$$|g| < (1+\xi^2)^{-r/2}\ N_r[g] < N_r[g]$$

we obtain

$$\tfrac{1}{2}|fg_1 + f_1 g| < (1+\xi^2)^{-r/2}\ N_r[f]N_r[g]$$

and conclude that

$$\nu\Gamma_1 < (1+\xi^2)^{-r/2}\ \alpha_o \nu N_r[f]N_r[g]$$

or

$$N_r[\Gamma_1| < \alpha_o N_r[f]N_r[g] \quad . \tag{A.8}$$

To estimate $\Gamma_2$, we transform the integral as in Ref.[9] (here we use $R(v,w)$ for the kernel $Q(v,w)$ in Ref. [9])

$$\nu\Gamma_2 = \int \tfrac{1}{2}|f(\xi+v)g(\xi+w) + f(\xi+w)g(\xi+v)|\,\omega^{\frac{1}{2}}(\xi+v+w)\frac{R(v,w)}{v^2}\ dv\ dw. \tag{A.9}$$

We recall that w is integrated over the plane perpendicular to v, and then v is integrated over the full 3-dimensional space. Resolving $\xi$ into components $\xi_1$ and $\xi_2$ parallel to v and w respectively, we have

$$\omega^{\frac{1}{2}}(\xi+v+w) = \omega^{\frac{1}{2}}(\xi_1+v)\omega^{\frac{1}{2}}(\xi_2+w) \qquad (A.10)$$

From eq. (60) of Ref. [9] we have

$$\int \omega^{\frac{1}{2}}(\xi_2+w)R(v,w)dw < Q_0 v \ . \qquad (A.11)$$

From

$$\begin{aligned}[1+(\xi+v)^2][1+(\xi+w)^2] &> 1+(\xi+v)^2 + (\xi+w)^2 \\ &= 1+\xi^2 + (\xi+v+w)^2 \\ &> 1+\xi^2\end{aligned}$$

we obtain

$$|f(\xi+v)g(\xi+w)| < (1+\xi^2)^{-r/2}N_r[f]N_r[g] \ . \qquad (A.12)$$

Substituting (A.11) and (A.12) into (A.9), we find

$$v\Gamma_2 < Q_0(1+\xi^2)^{-r/2}N_r[f]N_r[g]\int \frac{1}{v}\,\omega^{\frac{1}{2}}(\xi_1+v)dv \ . \qquad (A.13)$$

The last integral is bounded (it attains its maximum, $4(2\pi)^{1/4}$, at $\xi_1 = 0$), thus we have

$$N_3[\Gamma_2] < \beta_0\, N_r[f]N_r[g]$$

$$\beta_0 = 4(2\pi)^{1/4}\, Q_0 \ .$$

Finally we have the desired result,

$$N_r[\Gamma] < \gamma_0\, N_r[f]N_r[g] \qquad (A.15)$$

where $\gamma_0 = \alpha_0 + \beta_0$. A trivial modification of the above arguments yields the companion estimate

$$\not{N}_r[\Gamma] < \gamma_0\, N_r[f]\, \not{N}_r[g] \ . \qquad (A.16)$$

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